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Bayesian Value-at-Risk for a Portfolio: Multi- and Univariate Approaches Using MSF-SBEKK Models

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Abstract

The s-period ahead Value-at-Risk (VaR) for a portfolio of dimension n is considered and its Bayesian analysis is discussed. The VaR assessment can be based either on the n-variate predictive distribution of future returns on individual assets, or on the univariate Bayesian model for the portfolio value (or the return on portfolio). In both cases Bayesian VaR takes into account parameter uncertainty and non-linear relationship between ordinary and logarithmic returns. In the case of a large portfolio, the applicability of the n-variate approach to Bayesian VaR depends on the form of the statistical model for asset prices. We use the n-variate type I MSF-SBEKK(1,1) volatility model proposed specially to cope with large n. We compare empirical results obtained using this multivariate approach and the much simpler univariate approach based on modelling volatility of the value of a given portfolio.

Keywords: Bayesian econometrics, risk analysis, multivariate GARCH processes, multivariate SV processes, hybrid SV-GARCH models

JEL Classification: C11, C22, C32, C53, G17

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1 Introduction

Investors calculate Value-at-Risk (VaR) for their portfolios, which are usually quite large. VaR means the loss (of the portfolio value) that would be reached or exceeded with a given probability α (usually 0.05 or smaller) over a certain time horizon (most often from 1 to 10 days). Despite theoretical discussions (see Artzner, Delbaen, Eber, Heath 1999), VaR has become the standard measure of market risk used both by financial institutions and by their regulators; see Engle and Manganelli (2004).

VaR is a characteristic of the distribution of the future portfolio value (conditional on historical data on asset prices) and is closely related to its left tail. In practice, this probability distribution is unknown and is replaced by a statistical (sampling) model, that is a family of probability distributions; the data are used to choose its most appropriate element, which leads to the estimate of VaR. More traditional approaches to the assessment of VaR are based on parametric statistical models (usually from the GARCH family), which describe the whole distribution of future returns. The recently popular CAViaR approach (based on quantile regression) directly focuses on the α-quantile modelled non-parametrically; see Engle and Manganelli (2004).

In this paper we discuss and compare VaR assessment based on multi- and univariate parametric models. Multivariate approach is much more difficult, as it explicitly takes into account the full conditional covariance structure of asset prices: individual volatilities and correlations. On the other hand, VaR requires only the distribution of the future value of the portfolio; it can be derived using a univariate model for the historical values of the portfolio. Such an univariate approach is much simpler, since it does not need specifying the covariance structure of the assets.

In our comparison we refer to parametric models and use the Bayesian statistical paradigm that unifies the theory and practice of VaR. Within this paradigm, the parametric sampling models together with prior distributions can be used as building blocks for the unique predictive distribution of the future portfolio value. The predictive distribution automatically takes into account uncertainty about the parameters of the statistical model used to describe historical data. Also, specification (model) uncertainty can easily be incorporated using Bayesian pooling ("model averaging"), not considered in this paper. The predictive Bayesian formulation of VaR will be called Bayesian VaR.

The focus on the (left) tail of the predictive distribution requires (as its building block) a statistical model that is capable of estimating and forecasting the chances of extreme or outlying observations. The practical usefulness of Bayesian VaR depends on particular models under consideration as well as on numerical methods used in analysing the predictive distribution. Most of multivariate specifications in financial econometrics either belong to the MGARCH (Multivariate GARCH) or MSV (Multivariate Stochastic Volatility) classes or are based on copulas; see Bauwens, Laurent, Rombouts (2006), Tsay (2005). These models are difficult to estimate; only a few of them could be practical tools for large portfolios. A solution to the problem of simple, parsimonious multivariate volatility modelling is a hybrid model proposed by

Osiewalski (2009); see also Osiewalski and Pajor (2009). This hybrid model is based on scalar BEKK (SBEKK) correlation structure and the simplest MSV specification, the Multiplicative Stochastic Factor (MSF) model. Here we use the MSF-SBEKK type I model for portfolios of dimension $n = 34$ and $n = 50$. In order to make the univariate model of portfolio value comparable to the n-variate model of individual assets, we consider the univariate specification obtained from the MSF-SBEKK one by taking $n = 1$.
In the next section we discuss basic notions and introduce notation. In section 3 we present the foundations of Bayesian VaR. Section 4 is devoted to our models proposed for the assessment of VaR. Sections 5 and 6 contain empirical results for portfolios of dimension 34 and 50, respectively. Section 7 concludes.

2 Portfolio VaR - concepts, notation, modelling approaches

Consider a portfolio kept at present time (T) and consisting of n assets; a_i denotes the number of units of asset i possessed now and $S_{t,i}$ is the price of asset i at time t ($S_{t,i} > 0$, $a_i > 0$ for $i = 1, \ldots, n$), thus $W_t = \sum_{i=1}^{n} a_i S_{t,i}$ is the time t value of this portfolio. The s-period return rate on the portfolio is:

$$R^*_{t:t+s} = \frac{(W_{t+s} - W_t)}{W_t} = \sum_{i=1}^{n} \omega_{t,i} R_{t:t+s,i},$$

where $R_{t:t+s,i} = \frac{(S_{t+s,i} - S_{t,i})}{S_{t,i}}$ is the s-period return rate on asset i and $\omega_{t,i} = \frac{a_i S_{t,i}}{W_t}$ is the share of asset i in the time t portfolio value. For most results $a_i > 0$ is not required (short sale is allowed), only $W_t > 0$ has to be assumed. Note that the sum of $\omega_{t,i}$ over the assets ($i = 1, \ldots, n$) is always 1 by construction.
Assume that we observe the n-variate time series of individual return rates for $t = 1, \ldots, T$ and we are interested in forecasting $R^*_{T:T+s}$, the s-period ahead return on the portfolio kept at time T. Forecasting $R^*_{T:T+s}$ is closely related to the definition of $VaR_{T:T+s}$, the s-period ahead Value-at-Risk of the portfolio. If Ψ_T denotes the current and past asset prices, then $VaR_{T:T+s}(\alpha)$ for a given probability level α is defined by the following equality:

$$Pr\{W_{T+s} \leq W_T - VaR_{T:T+s}(\alpha)|\Psi_T\} = \alpha, \tag{1}$$

which can be written as

$$Pr\left\{R^*_{T:T+s} \leq \frac{-VaR_{T:T+s}(\alpha)}{W_T}|\Psi_T\right\} = \alpha. \tag{2}$$

Under any continuous distribution, the relative s-period ahead Value-at-Risk (corresponding to some fixed, small α) is the absolute value of the α-quantile of

the conditional distribution of the s-period ahead return on the portfolio, given the current and past asset prices.
The ordinary return rates $R_{t:t+s,i} > -1$ are rarely used in statistical modelling of asset prices and returns. Instead, the logarithmic return rates $r_{t+1,i} = \ln\left(\frac{S_{t+1,i}}{S_{t,i}}\right) = \ln(R_{t:t+s,i}+1)$ are the quantities being modelled; they can take any real value and easily aggregate over time:

$$r_{t:t+s,i} = \ln(R_{t:t+s,i}+1) = \ln\left(\frac{S_{t+s,i}}{S_{t,i}}\right) = \sum_{j=1}^{s} \ln\left(\frac{S_{t+j,i}}{S_{t+j-1,i}}\right) = \sum_{j=1}^{s} r_{t+j,i}$$

Since $R_{t:t+s,i}+1 = \exp\left(\sum_{j=1}^{s} r_{t+j,i}\right)$ and $R^*_{t:t+s} = \sum_{i=1}^{n} \omega_{t,i} R_{t:t+s,i}$, we can rewrite (1) as

$$Pr\left\{-1 + \sum_{i=1}^{n} \omega_{T,i} \exp\left(\sum_{j=1}^{s} r_{T+j,i}\right) \le -\frac{VaR_{T:T+s}(\alpha)}{W_T} | \Psi_T\right\} = \alpha, \qquad (3)$$

i.e. the relative VaR is the absolute value of the α-quantile of some non-linear function of future logarithmic returns.
The usual linear approximation $\exp\left(\sum_{j=1}^{s} r_{t+j,i}\right) \approx 1 + \sum_{j=1}^{s} r_{t+j,i}$ can lead to serious errors, especially when s is so large that the s-period ahead return distribution is diffuse. Consider a simple example with just one asset ($n = 1$) and the Student t distribution with 4 degrees of freedom, $St(4)$, for $10r_{T:T+s}$ (that is, the 0.1 $St(4)$ distribution for $r_{T:T+s}$ itself). This distribution of r_t can be obtained from the $N\left(0, \tau^{-1}\right)$ distribution of r_t (given its precision τ) and the Gamma distribution of τ (with mean 10 and variance 50), representing rather low precision. In this case (3) is equivalent to $Pr\left\{St(4) \le 10\ln\left(1 - \frac{VaR_{T:T+s}(\alpha)}{W_T}\right)\right\} = \alpha$; true and approximate values of relative VaR are presented in Table 1. For small α, the true relative VaR can be overestimated quite substantially.
Conditioning on observed data and small-sample inference on non-linear functions of unobserved quantities are natural within the Bayesian approach to statistics. Therefore this approach is advocated for determining the s-period ahead VaR.

Table 1: Relative VaR for $r_{T:T+s}$ distributed as 0.1 $St(4)$

α	0.005	0.01	0.0125	0.025	0.05
approximate VaR	0.4604	0.3747	0.3495	0.2776	0.2132
true VaR	0.3690	0.3125	0.2950	0.2424	0.1920

3 Foundations of Bayesian VaR assessment

The sampling model, i.e. a family of probability distributions of the observables $\widetilde{y} \in \widetilde{Y} \subset \mathbb{R}^N$ indexed by some parameter $\theta \in \Theta \subset \mathbb{R}^K$, is the common starting point of both the sampling-theory and Bayesian parametric approaches to statistics. In financial applications $\widetilde{y}$ groups all the modelled logarithmic return rates, including the forecasted ones. The Bayesian model is defined as a joint distribution on the product of the sample and parameter spaces ($\widetilde{Y}$ and Θ). In terms of densities, it can be represented as

$$p\left(\widetilde{y}, \theta\right) = p\left(\widetilde{y}|\theta\right) p\left(\theta\right), \tag{4}$$

where $p\left(\widetilde{y}|\theta\right)$ is the sampling density and $p\left(\theta\right)$ is the prior density. As in the Bayesian approach the parameters are not fundamentally different from unobservable (latent) variables, $p\left(\theta\right)$ will represent the distribution of all parameters and latent variables, if the latter are present in the model. In order to cover prediction as well as parameter estimation, assume that $\widetilde{y} = (y, y_f)$, where $y \in Y$ represents observed return rates, $y_f \in Y_f$ denotes unobserved returns (to be forecasted), and $\widetilde{Y} = Y \times Y_f$. Bayesian inference relies on the following decomposition of the joint density (4):

$$p\left(y, y_f, \theta\right) = p\left(y_f|y, \theta\right) p\left(y|\theta\right) p\left(\theta\right) = p\left(y_f|y, \theta\right) p\left(\theta|y\right) p\left(y\right), \tag{5}$$

Inference on all unknown and unobserved quantities (parameters, latent variables and future observables) can be based on the joint posterior – predictive density function

$$p\left(\theta, y_f|y\right) = p\left(y_f|y, \theta\right) p\left(\theta|y\right), \tag{6}$$

where $p\left(y_f|y, \theta\right)$ is the sampling predictive density (conditional on the parameters and latent variables), $p\left(\theta|y\right) = \frac{p(y|\theta)p(\theta)}{p(y)}$ is the posterior density (of the parameters and latent variables) and $p\left(y\right) = \int_{\Theta} p\left(y|\theta\right) p\left(\theta\right)\, d\theta$ is the marginal density of the observed returns.

If we are only interested in prediction of future returns, as in the case of determining the portfolio VaR through (3), we use the Bayesian predictive distribution

$$p\left(y_f|y\right) = \int_{\Theta} p\left(y_f|y, \theta\right) p\left(\theta|y\right)\, d\theta, \tag{7}$$

which fully reflects uncertainty regarding θ, given the data, the choice of a sampling model and a prior density; this uncertainty is formalized through the posterior density. If a particular function of y_f is of interest (like $R^*_{T:T+s}$, the s-period ahead portfolio return), its distribution is directly obtained from $p\left(y_f|y\right)$.

Non-Bayesian VaR assessments can be based on the sampling predictive distribution $p\left(y_f|y, \theta\right)$ with the parameters replaced by their estimates. The use of $p\left(y_f|y, \theta = \widehat{\theta}\right)$ can lead to substantially different inference on tail behaviour than relying on the

Bayesian predictive distribution. For a simple example assume that $n = 1$ and the sampling predictive density for future logarithmic returns, $p\left(r_{T:T+s}|y,\theta\right)$, is Normal with mean 0 and unknown precision τ, which has the Gamma posterior distribution with shape and scale parameter $\frac{\nu}{2}$; then $p\left(r_{T:T+s}|y\right)$ is Student t with ν degrees of freedom. In this case the usual non-Bayesian VaR would be calculated using the thin Normal tail and the Bayesian VaR would be based on the thicker Student tail, properly reflecting parameter uncertainty. Of course, there is little practical difference between both approaches when τ is estimated very precisely (large ν), but this need not be the case (like when ν is small, which leads to substantial differences).
Whereas the sampling-theory justification of inference procedures is based on the sampling properties in $\widetilde{Y}$ (given unknown, but fixed, parameter value θ), Bayesians consider the probability distribution of θ and y_f given the observed values of y, without contemplating what could have been observed in repeated sampling. On the formal level, introducing a distribution over the parameter space and conditioning on the observations are the distinctive features of the Bayesian approach. Also, the subjective interpretation of probability as a measure of degree of belief (or uncertainty) is widely adopted by Bayesian statisticians. Thus, the portfolio VaR fulfilling (1)-(3) can be interpreted in an intuitively straightforward manner: "given the data, the statistical model and prior information, one can be (1-α)·100% sure that the future value of a given portfolio, W_{T+s}, will be greater than $W_T - VaR_{T:T+s}(\alpha)$."
Finally, let us consider two modelling strategies for assessing portfolio VaR. The first one amounts to assuming some n-variate model for individual logarithmic returns $r_{t,i}$ and obtaining the α-quantile of the predictive distribution of

$$R^*_{T:T+s} = -1 + \sum_{i=1}^{n} \omega_{T,i} \exp\left(\sum_{j=1}^{s} r_{T+j,i}\right),$$

a non-linear function of future returns. The second approach amounts to directly modelling univariate series of portfolio logarithmic returns $r^W_{t+1} = \ln\left(\frac{W_{t+1}}{W_t}\right)$ and examining the predictive distribution of $r^W_{T:T+s} = \ln\left(\frac{W_{T+s}}{W_T}\right)$. Since

$$r^W_{t+1} = \ln\left(\sum_{i=1}^{n} \omega_{t,i} \exp\left(r_{t+1,i}\right)\right),$$

the univariate model that would exactly correspond to the n-variate specification is overly complicated and the only practical solution is to consider some standard univariate class for portfolio returns. Thus, the two approaches (n- and univariate) are not formally coherent and their comparison is an empirical question, addressed in this paper. Our conjecture is that a univariate model from a flexible parametric family can explain and predict portfolio returns not worse than any n-variate specification that requires huge simplifications in order to cope with large n.

4 The hybrid VAR(1)-MSF-SBEKK type I Bayesian model

First we consider a multivariate specification for individual assets. Let $r_t = (r_{t,1} \dots r_{t,n})$ denote n-variate observations on logarithmic return rates, which we model using the basic VAR(1) framework:

$$r_t = \delta_0 + r_{t-1}\Delta + \varepsilon_t,\ t = 1, \dots, T, \dots, T+s. \tag{8}$$

The $n(n+1)$ elements of $\delta = (\delta_0\ (\mathrm{vec}\Delta)')'$ are common parameters, which can be treated as *a priori* independent of all other (model-specific) parameters; we can assume for them some multivariate prior, e.g. standard Normal $N\left(0, I_{n(n+1)}\right)$, truncated by the restriction that all eigenvalues of Δ lie inside the unit circle.
Following Osiewalski and Pajor (2009), we specify the conditional distribution of the residual process ε_t by conditioning on its past Ψ_{t-1}, some univariate latent process (g_t) and the parameters. We assume the so-called type I hybrid specification:

$$\varepsilon_t = \zeta_t H_t^{\frac{1}{2}} \sqrt{g_t}, \tag{9}$$

$$\ln g_t = \phi \ln g_{t-1} + \sigma_g \eta_t,\ \ (\zeta_t, \eta_t)' \sim iiN(0_{[(n+1)\times 1]}, I_{n+1}), \tag{10}$$

$$H_t = (1-\beta-\gamma)\, A + \beta \left(\varepsilon_{t-1}' \varepsilon_{t-1}\right) + \gamma H_{t-1}. \tag{11}$$

That is, ε_t is conditionally Normal with mean vector 0 and covariance matrix $g_t H_t$, where g_t is a latent process and H_t is a square matrix of order n that has the scalar BEKK(1,1) structure. Thus, the corresponding conditional distribution of r_t (given its past and latent variables) is Normal with mean $\mu_t = \delta_0 + r_{t-1}\Delta$ and covariance matrix $g_t H_t$.
The presence of the latent AR(1) process in the conditional covariance matrix helps in explaining outlying observations, and the dependence on the past data (through the SBEKK structure of H_t) prevents the entries of the conditional covariance matrix $g_t H_t$ from sharing the same dynamic pattern. Thus the model has time-varying conditional correlations without introducing more latent processes. In fact, the hybrid model defined by (9)-(11) nests two simple basic structures. In the limiting case when $\sigma_g \to 0$ and $\phi = 0$ we are in the SBEKK model, while $\beta = 0$ and $\gamma = 0$ lead to the MSF case.
In (11) A is a free symmetric positive definite matrix of order n; for A^{-1} we assume the Wishart prior with n degrees of freedom and mean I_n; β and γ are free scalar parameters, jointly uniformly distributed over the unit simplex. As regards initial conditions for H_t, we can either take $H_0 = h_0 I_n$ and treat $h_0 > 0$ as an additional parameter, *a priori* Exponentially distributed with mean 1, or fix H_0. For the parameters of the latent process we use the same priors as Pajor (2005); for ϕ: Normal with mean 0 and variance 100, truncated to (-1, 1), for σ_g^{-2}: Exponential with mean 200; g_0 is fixed (equals 1).

In order to obtain the required quantiles of the predictive distribution of future logarithmic returns, we follow the approximation explained in Osiewalski and Pajor (2009). That is, we use OLS for the VAR(1) parameters and replace A by the empirical covariance matrix of the OLS residuals from the VAR(1) part. The Bayesian analysis for the remaining parameters and future return rates is then based on the conditional posterior and predictive distributions given the particular values of the highly dimensional parameters (δ and A). These conditional distributions are sampled using the Gibbs scheme with Metropolis-Hastings steps, as shown in detail in Osiewalski and Pajor (2009).

In order to make the univariate model of portfolio value comparable to the n-variate volatility model of individual assets, we consider for the portfolio logarithmic returns r_t^W the univariate AR(1) specification with the error term described by the hybrid SV-GARCH(1,1) process, which is the $n = 1$ special case of the MSF-SBEKK structure. So we assume

$$r_t^W = \delta_0^* + \delta^* r_{t-1}^* + \varepsilon_t^*, \tag{12}$$

$$\varepsilon_t^* = \zeta_t^* \sqrt{g_t h_t}, \tag{13}$$

$$\ln g_t = \phi \ln g_{t-1} + \sigma_g \eta_t, \;\; (\zeta_t^*, \eta_t)' \sim iiN(0_{[2\times 1]}, I_2), \tag{14}$$

$$h_t = (1-\beta-\gamma)\, a^* + \beta \left(\varepsilon_{t-1}^*\right)^2 + \gamma h_{t-1}, \; t = 1, \ldots, T, \ldots, T+s. \tag{15}$$

We take the prior distribution corresponding to the previous (n-variate) case (with $n = 1$). Now we do not face the dimensionality problem, but for comparison with the n-variate model, the posterior and predictive distribution is sampled (using the Gibbs scheme with Metropolis-Hastings steps) conditionally on preliminary non-Bayesian estimates as in the n-variate case.

5 VaR for a portfolio with 34 assets

As the first dataset we use the same stock data representing 34 companies, which are used in Osiewalski and Pajor (2009). Summary statistics for the percentage daily logarithmic returns ($100r_{t,i}$) in the period January 30, 2003 – August 29, 2007 are shown in Table A1 in Appendix; on August 29, 2007 companies number 1–23 were included in mWIG40 and number 24–34 in WIG20, two important indices of the Warsaw Stock Exchange. The approximate Bayesian approach (using the proposed data-based values of the highly dimensional matrix parameters) was applied. The posterior results on volatility and conditional correlation are presented in Osiewalski and Pajor (2009) for the whole length of time series ($T = 1149$). Here we start with $T = 939$ initial observations (covering the period February 3, 2003 – October 23, 2006) and consider $p = 200$ VaR assessments for 1-, 2-, …, 10-day trading horizons. For Bayesian estimation the whole dataset available at time $T+k$ ($k = 0, 1, \ldots, p-1$) is used. We calculate predictive distributions of r_t (or r_t^W) based on the dataset available at time $T+k$ for each $k = 0, 1, \ldots, p-1$ (up to $T+p-1 = 1138$). Thus

we obtained 200 predictive distributions for 1-, 2-, ..., 10-day forecast horizons, and then $VaR_{t:t+s}(\alpha)$ for $t = T, \ldots, T + p - 1$ and $s = 1, 2, \ldots, 10$.
Our portfolio consists of one unit of each asset, i.e. $a = (a_1, a_2, \ldots, a_n) = (1, \ldots, 1)'$. The univariate time series of the value of such portfolio is characterised by the daily logarithmic returns r_t^W presented in Figure 1; the daily value changes are shown in Figure 2. The $VaR_{t:t+1}(\alpha)$ assessments for $\alpha = 0.05$ and $\alpha = 0.1$ are presented in Figures 3 and 4, respectively.

Figure 1: Daily growth rates of the portfolio value; $n = 34$ and $a = (1, \ldots, 1)'$ (January 31, 2003 – August 28, 2007); the vertical line represents October 23, 2006

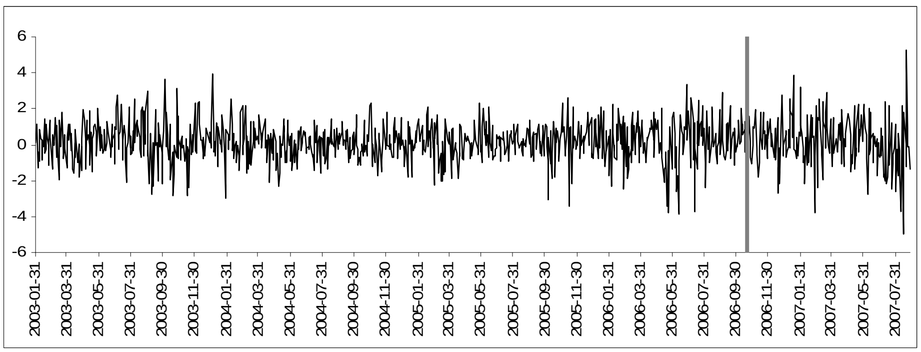
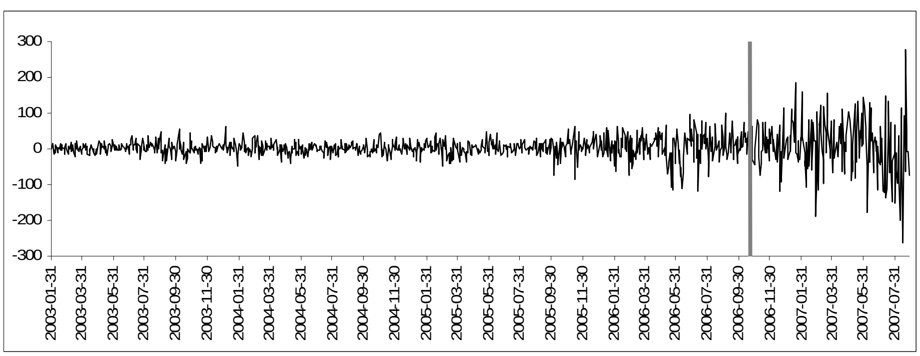


Figure 2: Daily changes in the portfolio value (January 31, 2003 – August 28, 2007; $n = 34$); the vertical line represents October 23, 2006



In order to compare 1-day ahead Value-at-Risk obtained in two different ways, i.e. using n-variate MSF-SBEKK model for individual assets or its univariate counterpart for the portfolio value, we use popular non-Bayesian criteria. They include: the failure

Figure 3: $-VaR_{t:t+1}(0.05)$, $n = 34$

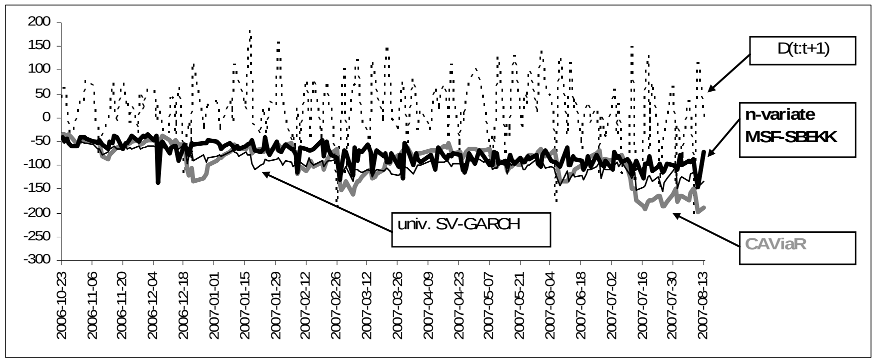
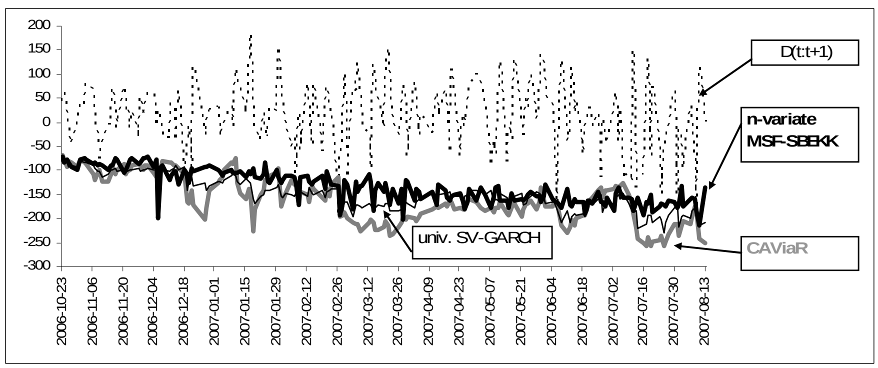


Figure 4: $-VaR_{t:t+1}(0.01)$, $n = 34$



rate and p-value for the Kupiec test as well as different loss functions (defined below) for $VaR_{t:t+1}(\alpha)$; see Tables 2–4. We also use the Conditional Autoregressive Value at Risk (or CAViaR) model (with asymmetric slope):

$$q_t(\alpha) = \beta_0 + \beta_1 q_{t-1}(\alpha) + \beta_2 |D_{t-2:t-1}| + \beta_3 |D_{t-2:t-1}| I_{(-\infty,0)}(D_{t-2:t-1}) \qquad (16)$$

of Engle and Manganelli (2004); it is applied directly to the series $\{D_{t:t+s}\}$ of daily value changes $D_{t:t+s} = W_{t+s} - W_t$ (not to the logarithmic returns); thus, $q_t(\alpha)$ denotes the conditional α-quantile of $D_{t-1:t}$, $I_{(-\infty,0)}(\cdot)$ is the characteristic function of the interval $(-\infty, 0)$.

The losses are generally calculated as $L_s = \frac{1}{p} \sum_{t=T}^{T+p-1} l_{t:t+s}$, where for $l_{t:t+s}$ we have the "tick" loss if:

$$l_{t:t+s} = \begin{cases} (\alpha - 1)\left(D_{t:t+s} + VaR_{t:t+s}(\alpha)\right), & if\ D_{t:t+s} < -VaR_{t:t+s}(\alpha), \\ \alpha\left(D_{t:t+s} + VaR_{t:t+s}(\alpha)\right), & if\ D_{t:t+s} \geq -VaR_{t:t+s}(\alpha); \end{cases}$$

the Lopez loss if:

$$l_{t:t+s} = \begin{cases} 1 + \left(D_{t:t+s} + VaR_{t:t+s}(\alpha)\right)^2, & if\ D_{t:t+s} < -VaR_{t:t+s}(\alpha), \\ 0, & if\ D_{t:t+s} \geq -VaR_{t:t+s}(\alpha); \end{cases}$$

the firm's loss if:

$$l_{t:t+s} = \begin{cases} \left(D_{t:t+s} + VaR_{t:t+s}(\alpha)\right)^2, & if\ D_{t:t+s} < -VaR_{t:t+s}(\alpha), \\ cVaR_{t:t+s}(\alpha), & if\ D_{t:t+s} \geq -VaR_{t:t+s}(\alpha). \end{cases}$$

see e.g. Lopez (1998), Sarma, Thomas, Shah (2003), Lee (2008). We also compute (and present in Table 4) the average loss on the portfolio when the loss is larger than $VaR_{t:t+s}(\alpha)$, that is

$$AL_s = \frac{\sum_{t=T}^{T+p-1} I_{(-\infty,0)}\left(D_{t:t+s} + VaR_{t:t+s}(\alpha)\right)\left|D_{t:t+s}\right|}{\sum_{t=T}^{T+p-1} I_{(-\infty,0)}\left(D_{t:t+s} + VaR_{t:t+s}(\alpha)\right)}$$

The outcomes of the Kupiec test for the 1-day ahead VaR seem to indicate that the univariate approach is more accurate and our Bayesian assessment competes with the one based on CAViaR (Table 2; the best case is in bold). The "tick" loss function does not give such a clear picture, but the Lopez and firms' losses are smallest for our Bayesian VaR based on the univariate approach (Table 3 and 4). The results show that, in the case of this particular portfolio, the n-variate MSF-SBEKK approach is unnecessary for risk assessment. On the other hand, the univariate special case gives us the flexible parametric SV-GARCH(1,1) specification that can be very successful in VaR analysis. It is usually not worse than CAViaR (sometimes much better) and leads to assessments that are highly correlated with the ones based on CAViaR; see Table 5.

In Tables 6 and 7 we present $VaR_{t:t+s}(\alpha)$ results for all forecast horizons ($s = 1, 2, \ldots, 10$); the results were obtained using univariate and n-variate MSF-SBEKK models, respectively. The univariate SV-GARCH model gives better VaR forecasts for all s.

It may be the case that the approximate character of our posterior and predictive analysis, based on the OLS estimates of matrix parameters, is partly responsible for the poor performance of our n-variate model. However, this is impossible to verify as the exact posterior analysis is infeasible for $n = 34$.

Table 2: The failure rate and p-value for the Kupiec test for $VaR_{t:t+1}(\alpha)$, $n = 34$

α	(frequency) failure rate			Kupiec test p-value		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	0.02	**0.015**	0.020	0.211	**0.508**	0.211
0.025	0.06	**0.035**	0.040	0.007	**0.392**	0.211
0.05	0.1	0.075	**0.065**	0.004	0.130	**0.351**
0.1	0.185	**0.105**	0.135	0.000	**0.815**	0.115

Note: The failure rate is defined as the proportion of $D_{t:t+1}$'s smaller than the $-VaR_{t:t+1}(\alpha)$

Table 3: "Tick" and Lopez loss functions for $VaR_{t:t+1}(\alpha)$, $n = 34$

α	"Tick" loss function			Lopez loss function		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	**1.92**	2.06	2.239	19.18	**17.97**	29.941
0.025	4.47	**4.36**	4.641	85.10	**64.09**	81.045
0.05	8.05	7.60	**7.487**	221.61	**145.63**	181.945
0.1	13.51	**12.42**	12.486	509.19	**324.67**	361.133

Table 4: Firm's loss functions for $VaR_{t:t+1}(\alpha)$ and average loss on the portfolio when the loss is larger than $VaR_{t:t+1}(\alpha)$, $n = 34$

α	Firm's loss function with $c = 0.000167$ (average WIBOR O/N rate)			Average loss on the portfolio when the loss is larger than $VaR_{t:t+1}(\alpha)$		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	19.178	**17.979**	29.947	170.848	168.680	170.848
0.025	85.052	**64.076**	81.025	133.508	149.227	136.856
0.05	221.522	**145.575**	181.894	119.272	129.937	115.975
0.1	509.010	**324.577**	361.007	93.605	118.261	100.464

Table 5: Correlation coefficients between $VaR_{t:t+1}(\alpha)$ for $\alpha = 0.01$ and $\alpha = 0.05$ (upper part), for $\alpha = 0.025$ and $\alpha = 0.1$ (lower part), $n = 34$

α=0.01 α=0.025	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	α=0.05 α=0.1	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
n-variate MSF-SBEKK	1	0.804	0.692	n-variate MSF-SBEKK	1	0.745	0.553
univariate SV-GARCH	0.776	1	**0.899**	univariate SV-GARCH	0.684	1	**0.806**
CAViaR	0.614	**0.875**	1	CAViaR	0.503	**0.797**	1

Table 6: $VaR_{t:t+s}(0.05)$ - univariate MSF-SBEKK (SV-GARCH), $n = 34$

s	1	2	3	4	5	6	7	8	9	10
FR	0.075	0.075	0.08	0.095	0.09	0.08	0.08	0.075	0.08	0.1
p-value for Kupiec test	**0.1296**	**0.1296**	**0.0722**	**0.009**	**0.019**	**0.0722**	**0.0722**	**0.1296**	**0.0722**	**0.004**
AL_s	129.94	205.72	223.99	234.91	269.29	309.95	352.62	372.33	428.43	433.15
tick loss	**7.6035**	**11.864**	**14.104**	**15.477**	**17.8**	**19.625**	**21.411**	**23.41**	**26.408**	**28.184**
Lopez loss	**1.3567**	**1.4166**	**1.3658**	**1.2836**	**1.3085**	**1.3539**	**1.3495**	**1.3872**	**1.4219**	**1.3519**

Table 7: $VaR_{t:t+s}(0.05)$ – n-variate MSF-SBEKK, $n = 34$

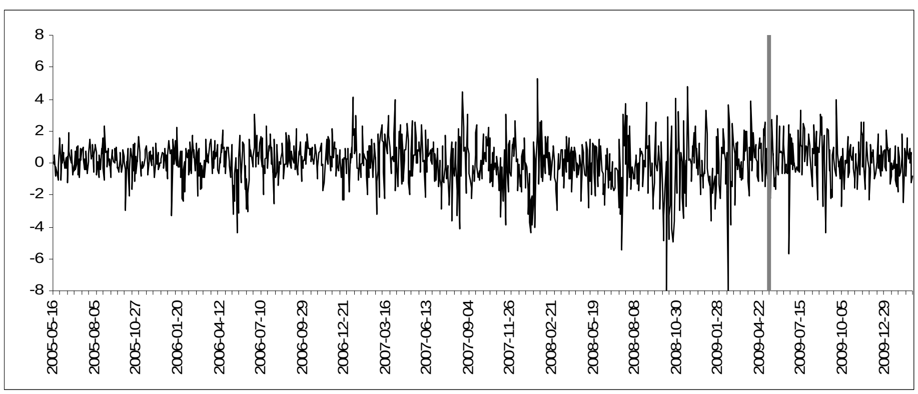
s	1	2	3	4	5	6	7	8	9	10
FR	0.1	0.09	0.115	0.135	0.135	0.135	0.13	0.14	0.12	0.135
p-value for Kupiec test	0.004	0.019	0.0003	$4 \cdot 10^{-6}$	$4 \cdot 10^{-6}$	$4 \cdot 10^{-6}$	10^{-6}	10^{-6}	0.0001	$4 \cdot 10^{-6}$
AL_s	119.27	194.72	202.63	220.73	245.24	281.7	313.44	325.8	388.05	398.16
tick loss	8.0514	13.438	15.295	17.546	19.886	22.623	25.379	27.841	31.546	35.001
Lopez loss	221.61	748.03	843.22	872.75	1266.7	1775.7	2280.1	3107.9	3891.2	4352.6

6 VaR for a portfolio with 50 assets

6.1 One unit of each asset

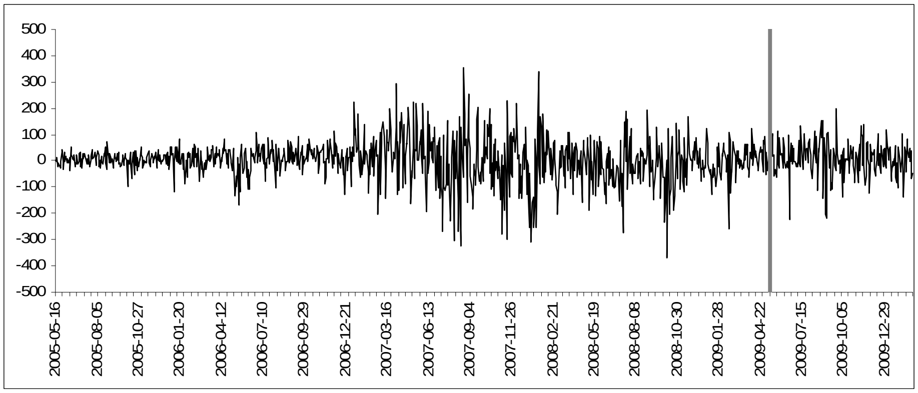
Now we use stock data (on 50 companies) from the period May 13, 2005 – February, 23, 2010 ($T = 1149$); in February or March 2010 companies number 1–34 were included in mWIG40 and 35–50 in WIG20. Summary statistics for the daily percentage logarithmic returns ($100r_{t,i}$) are shown in Table A2 in Appendix. Again, the considered portfolio consists of one unit of each asset. The percentage logarithmic returns and daily changes of the portfolio value are shown in Figures 5 and 6, respectively. While the previous time series (of the same length) ended just before the financial crisis, now we analyse the data that include the whole period of market turbulences. So there are two new aspects: the financial crisis and a larger portfolio ($n = 50$). For the n-variate model we use the same approximate Bayesian approach as previously. Again, we start with $T = 998$ initial observations (now from the period May 13, 2005 – May, 12, 2009) and consider $p = 200$ VaR assessments for 1-, 2- ,..., 10-day trading horizons. Note that our analysis covers the period of a slow recovery from the very deep crisis.

Figure 5: Daily growth rates of the portfolio value; $n = 50$ and $a = (1, \ldots, 1)'$ (May 16, 2005 – February 23, 2010); the vertical line represents May 12, 2009



The results presented for one day ahead VaR (Tables 8-10) clearly indicate that now the n-variate approach is more accurate and that our Bayesian assessment (based on the parametric MSF-SBEKK structure) competes with the one based on CAViaR. Interestingly, VaRs based on univariate approaches (CAViaR and SV-GARCH) are highly correlated, as in the previous example; see Table 11. For this dataset the s-day ahead VaR for $s > 1$, obtained within the n-variate model, is worse (than the assessment based on the univariate SV-GARCH model) only with respect to the tick

Figure 6: Daily changes in the portfolio value (May 16, 2005 – February 23, 2010; $n = 50$; $a = (1, \ldots, 1)'$); the vertical line represents May 12, 2009



loss; it is usually much better in terms of the failure rate and Lopez loss (see Tables 12 and 13).

Table 8: The failure rate and p-value for the Kupiec test for $VaR_{t:t+1}(\alpha)$, $n = 50$, $a = (1, \ldots, 1)'$

α	(frequency) failure rate			Kupiec test p-value		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	**0.010**	0.015	0.015	**1.000**	0.508	0.508
0.025	0.020	**0.025**	0.03	0.639	**1.000**	0.660
0.05	**0.045**	0.065	0.06	**0.742**	0.351	0.529
0.1	0.110	0.12	**0.105**	0.642	0.359	**0.815**

Note: The failure rate is defined as the proportion of $D_{t:t+1}$'s smaller than the $-VaR_{t:t+1}(\alpha)$

Table 9: "Tick" and Lopez loss functions for $VaR_{t:t+1}(\alpha)$, $n = 50$, $a = (1, \ldots, 1)'$

α	"Tick" loss function			Lopez loss function		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	**2.286**	2.341	2.430	63.316	68.848	**19.523**
0.025	**4.389**	4.509	4.902	112.060	133.847	**99.808**
0.05	**7.431**	7.581	8.018	**187.108**	227.417	210.195
0.1	**12.324**	12.427	12.920	**348.888**	416.769	366.585

Table 10: Firm's loss functions for $VaR_{t:t+1}(\alpha)$ and average loss on the portfolio when the loss is larger than $VaR_{t:t+1}(\alpha)$, $n = 50$ and $a = (1, \ldots, 1)'$

α	Firm's loss function with $c = 0.000114$ (average WIBOR O/N rate)			Average loss on the portfolio when the loss is larger than $VaR_{t:t+1}(\alpha)$		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	63.325	68.850	**19.529**	213.700	188.880	215.610
0.025	112.055	133.835	**99.793**	177.075	169.632	159.928
0.05	**187.074**	227.361	210.146	146.729	133.258	135.460
0.1	**348.786**	416.656	366.488	109.687	110.548	112.785

Figure 7: $-VaR_{t:t+1}(0.05)$ for $n = 50$, $a = (1, \ldots, 1)'$

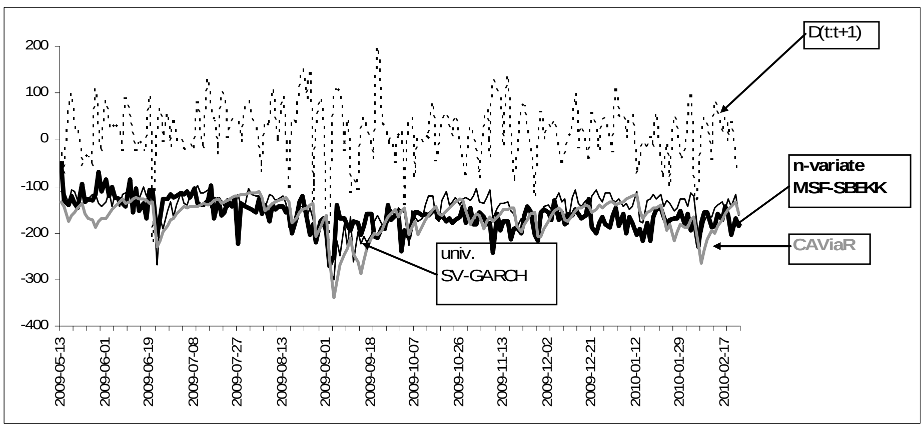


Table 11: Correlation coefficients between $VaR_{t:t+1}(\alpha)$ for $\alpha = 0.01$ and $\alpha = 0.05$ (upper part), for $\alpha = 0.025$ and $\alpha = 0.1$ (lower part), $n = 50$, $a = (1, \ldots, 1)'$

α=0.01 α=0.025	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	α=0.05 α=0.1	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
n-variate MSF-SBEKK	1	0.446	0.326	n-variate MSF-SBEKK	1	0.468	0.361
univariate SV-GARCH	0.452	1	**0.689**	univariate SV-GARCH	0.481	1	**0.808**
CAViaR	0.371	**0.795**	1	CAViaR	0.329	**0.783**	1

Figure 8: $-VaR_{t:t+1}(0.01)$ for $n = 50$, $a = (1, \ldots, 1)'$

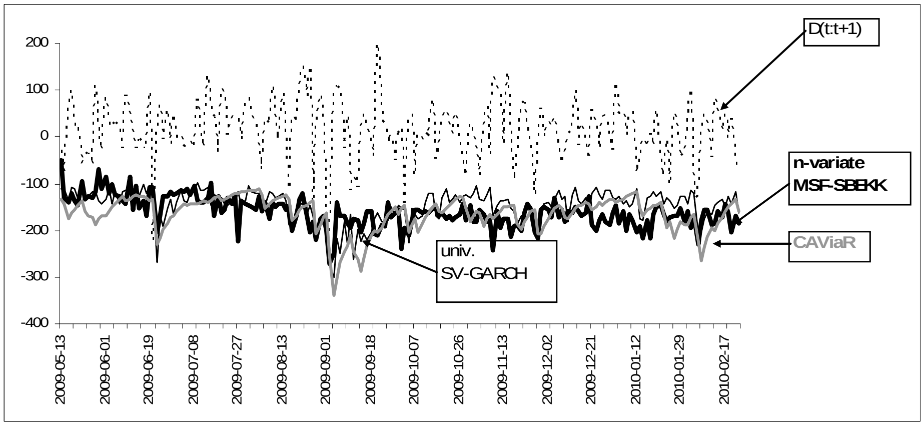


Table 12: $VaR_{t:t+s}(0.05)$ - univariate MSF-SBEKK (SV-GARCH), $n = 50$, $a = (1, \ldots, 1)'$

s	1	2	3	4	5	6	7	8	9	10
FR	0.1	0.075	0.07	0.085	0.065	0.075	0.09	0.07	0.065	0.085
p-value for Kupiec test	0.00	0.13	0.22	0.04	**0.35**	0.13	0.02	0.22	0.35	0.04
AL_s	12.73	14.67	18.77	24.83	22.70	29.35	35.88	31.31	30.64	40.81
tick loss	7.57	**9.99**	**12.88**	**14.64**	**17.16**	19.49	**19.56**	21.37	**22.94**	**24.07**
Lopez loss	145.08	206.46	416.52	538.06	752.14	957.59	690.62	859.62	1140.31	1018.98

Table 13: $VaR_{t:t+s}(0.05)$ – n-variate MSF-SBEKK, $n = 50$, $a = (1, \ldots, 1)'$

s	1	2	3	4	5	6	7	8	9	10
FR	0.065	0.045	0.050	0.035	0.035	0.040	0.045	0.055	0.040	0.040
p-value for Kupiec test	**0.351**	**0.742**	**1.000**	**0.305**	0.305	**0.502**	**0.742**	**0.749**	**0.502**	**0.502**
AL_s	8.21	9.70	14.06	12.13	13.87	16.55	19.20	24.08	19.97	19.79
tick loss	**7.51**	10.66	13.77	15.15	17.90	**18.93**	19.68	**21.21**	23.34	24.35
Lopez loss	114.03	160.67	274.67	263.37	449.77	405.15	127.13	144.04	284.96	313.36

6.2 Comparable shares of assets

Now we use the same stock data as previously, but the considered portfolio consists of $a_i = a_{\tau,i} = \frac{\frac{1}{n}\sum_{i=1}^{n} S_{\tau,i}}{S_{\tau,i}}$ units of asset i, that is $\omega_{\tau,i} = \frac{1}{50}$, where $i = 1, \ldots, 50$, and τ represents May 12, 2009. (The values of a_i are presented in the last column of Table

A2.) Of course, the shares $\omega_{\tau,i}$ vary over time, but they are more balanced than in the previous case (with one unit of each asset). The logarithmic returns and daily changes of the portfolio value are shown in Figures 9 and 10, respectively. Again, we start with $T = 998$ initial observations (from the period May 13, 2005 – May, 12, 2009) and consider $p = 200$ VaR assessments for 1-,2-,. . . , 10-day trading horizons.
The results for 1-day ahead VaR (Tables 14–16) do not lead to simple conclusions. Again, the univariate SV-GARCH model gives VAR assessments that are highly correlated with the ones based on CAViaR (Table 17). Which model is better depends on the particular criterion. For example, the "tick" loss indicates some preference for the n-variate MSF-SBEKK model, while the Lopez and firm's loss suggest that CAViaR is the optimal model.

Figure 9: Daily growth rates of the portfolio value; $n = 50$ and $\omega_{\tau,i} = \frac{1}{50}$ (May 16, 2005 – February 23, 2010); the vertical line represents May 12, 2009

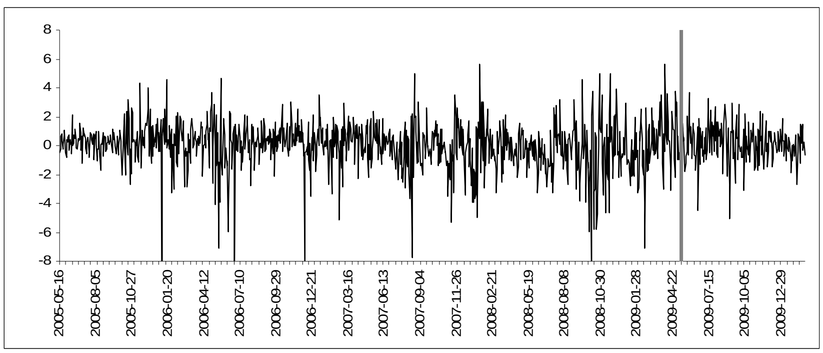
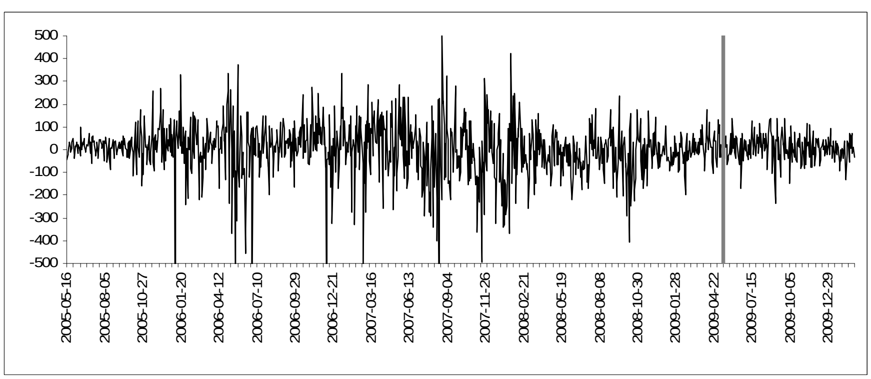


Figure 10: Daily changes in the portfolio value (May 16, 2005 – February 23, 2010; $n = 50$; $\omega_{\tau,i} = \frac{1}{50}$); the vertical line represents May 12, 2009



As previously, we also consider the s-day ahead VaR for $s > 1$. Again, according to the Lopez loss criterion, the n-variate MSF-SBEKK model is better than its univariate counterpart (the SV-GARCH model); the latter becomes important if we focus on the tick loss for $s > 6$ and Kupiec test for $s > 2$ (see Tables 18 and 19).
Note that the empirical findings obtained for the porftolio with balanced shares are not very similar to the previous ones, based on the portfolio with one unit of each asset. And both are different from the outcomes for the porfolio in Section 5 ($n = 34$), so any generalisation of our empirical results is hardly possible.
Finally, in Table 20 we present the posterior means and standard deviations, based on the whole time series, for basic MSF-SBEKK parameters (given the OLS estimates of the remaining parameters); we also show the results for the previous dataset ($n = 34$). The approximate posterior moments in n-variate models are very similar for the two datasets, but their counterparts in univariate SV-GARCH models are different between the datasets and portfolios (and from the n-variate cases) and show that the SV part is crucial.

Table 14: The failure rate and p-value for the Kupiec test for $VaR_{t:t+1}(\alpha)$, $n = 50$, $\omega_{\tau,i} = \frac{1}{50}$

α	(frequency) failure rate			Kupiec test p-value		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	**0.01**	0.02	**0.01**	**1.000**	0.211	**1.000**
0.025	0.02	**0.025**	**0.025**	0.639	**1.000**	**1.000**
0.05	**0.055**	0.04	0.035	**0.749**	0.502	0.305
0.1	0.09	**0.1**	0.09	0.632	**1.000**	0.632

Note: The failure rate is defined as the proportion of $D_{t:t+1}$'s smaller than the $-VaR_{t:t+1}(\alpha)$

Table 15: "Tick" and Lopez loss functions for $VaR_{t:t+1}(\alpha)$, $n = 50$ and $\omega_{\tau,i} = \frac{1}{50}$

α	"Tick" loss function			Lopez loss function		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	1.991	2.084	**1.971**	20.691	18.022	**6.400**
0.025	**3.942**	4.258	4.210	52.341	71.158	**13.531**
0.05	**6.729**	6.894	7.211	106.337	146.965	**85.306**
0.1	11.139	**10.996**	11.464	241.865	284.189	**228.027**

Table 16: Firm's loss functions for $VaR_{t:t+1}(\alpha)$ and average loss on the portfolio when the loss is larger than $VaR_{t:t+1}(\alpha)$, $n = 50$ and $\omega_{\tau,i} = \frac{1}{50}$

α	Firm's loss function with $c = 0.000114$ (average WIBOR O/N rate)			Average loss on the portfolio when the loss is larger than $VaR_{t:t+1}(\alpha)$		
	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
0.01	20.698	18.019	**6.409**	125.351	172.077	151.327
0.025	52.334	71.147	**13.522**	155.182	156.248	155.489
0.05	106.293	146.936	**85.283**	122.419	141.029	143.071
0.1	241.782	284.096	**227.945**	104.454	102.321	106.855

Figure 11: $-VaR_{t:t+1}(0.05)$ for $n = 50$, $\omega_{\tau,i} = \frac{1}{50}$

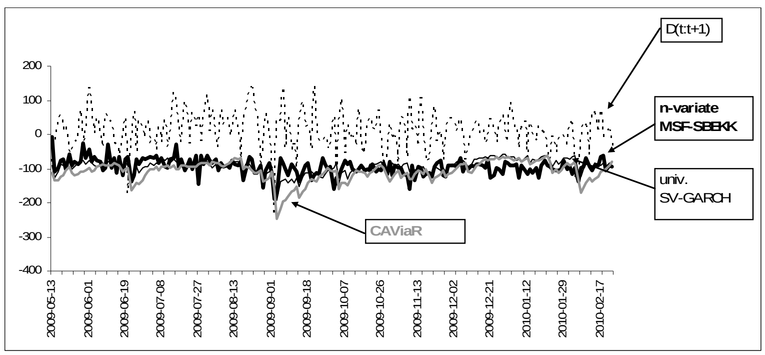


Figure 12: $-VaR_{t:t+1}(0.01)$ for $n = 50$, $\omega_{\tau,i} = \frac{1}{50}$

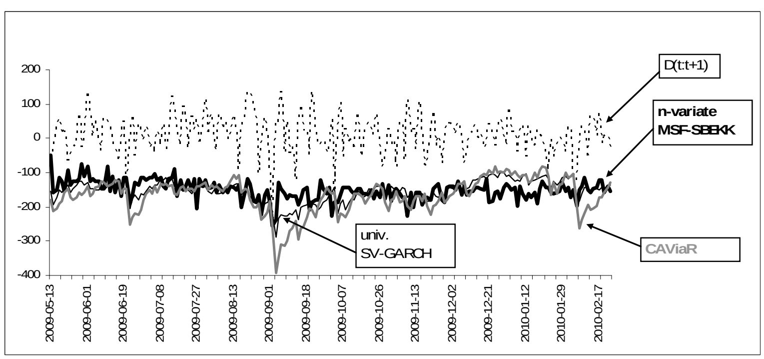


Table 17: Correlation coefficients between $VaR_{t:t+1}(\alpha)$ for $\alpha = 0.01$ and $\alpha = 0.05$ (upper part), for $\alpha = 0.025$ and $\alpha = 0.1$ (lower part), $n = 50$, $\omega_{\tau,i} = \frac{1}{50}$

α=0.01 α=0.025	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR	α=0.05 α=0.1	n-variate MSF-SBEKK	univariate SV-GARCH	CAViaR
n-variate MSF-SBEKK	1	0.362	0.264	n-variate MSF-SBEKK	1	0.372	0.247
univariate SV-GARCH	0.355	1	**0.902**	univariate SV-GARCH	0.402	1	**0.986**
CAViaR	0.276	**0.847**	1	CAViaR	0.247	**0.765**	1

Table 18: $VaR_{t:t+s}(0.05)$ - univariate MSF-SBEKK (SV-GARCH), $n = 50$, $\omega_{\tau,i} = \frac{1}{50}$

s	1	2	3	4	5	6	7	8	9	10
FR	0.040	0.030	0.040	0.040	0.045	0.040	0.035	0.005	0.010	0.005
p-value for Kupiec test	0.502	0.162	**0.502**	**0.502**	**0.742**	**0.502**	**0.305**	0.000	0.002	0.000
AL_s	141.03	203.01	228.68	249.28	230.08	266.19	283.42	255.71	279.58	321.78
tick loss	6.89	10.21	12.86	13.99	14.81	16.89	**17.38**	**18.20**	**19.59**	**20.87**
Lopez loss	146.97	385.70	356.18	147.67	52.09	133.13	25.03	6.83	3.20	2.19

Table 19: $VaR_{t:t+s}(0.05)$ – n-variate MSF-SBEKK, $n = 50$, $\omega_{\tau,i} = \frac{1}{50}$

s	1	2	3	4	5	6	7	8	9	10
FR	0.040	0.030	0.040	0.040	0.045	0.040	0.035	0.005	0.010	0.005
p-value for Kupiec test	**0.749**	**0.502**	0.305	0.074	0.008	0.002	0.008	0.000	0.000	0.000
AL_s	122.42	182.49	229.79	269.80	277.70	345.78	350.23	353.36	0	0
tick loss	**6.729**	**10.503**	**12.372**	**13.375**	**14.266**	**16.457**	17.801	18.506	19.708	20.962
Lopez loss	**106.34**	**329.20**	**258.96**	**70.42**	**4.87**	**108.37**	**42.37**	**1.54**	**0.00**	**0.00**

7 Concluding remarks

The aim of the paper was threefold. First, we wanted to compare the n-variate and univariate approaches to risk assessment for a large portfolio. Second, we were eager to learn how the new hybrid MSF-SBEKK type I specification would work in

Table 20: Posterior means (and standard deviations) of the main MSF-SBEKK parameters

Example	model	ϕ	σ_g^2	β	γ	$\beta+\gamma$
$n = 34$ $T = 1149$	n-variate	0.4995 (0.0339)	0.1874 (0.0113)	0.0167 (0.0014)	0.8523 (0.0180)	0.8690 (0.0168)
	univariate $a = (1, \dots, 1)$	0.9817 (0.0100)	0.0117 (0.0057)	0.0145 (0.0120)	0.3105 (0.1970)	0.3253 (0.1965)
$n = 50$ $T = 1198$	n-variate	0.5658 (0.0311)	0.1223 (0.0076)	0.0119 (0.0010)	0.8423 (0.0184)	0.8542 (0.0177)
	univariate $a = (1, \dots, 1)$	0.9576 (0.0334)	0.0351 (0.0196)	0.0178 (0.0159)	0.6650 (0.2927)	0.6828 (0.2869)
	univariate $\omega_{\tau,i} = \frac{1}{50}$	0.9621 (0.0171)	0.0479 (0.0177)	0.0084 (0.0075)	0.6331 (0.2949)	0.6425 (0.2934)

practice. Third, we wanted to show the merits of the Bayesian parametric approach to Value-at-Risk.

It is not clear that, for VaR assessment, univariate modelling (of portfolio value – instead of portfolio components) is enough as we initially (wrongly) conjectured. Multivariate specifications of asset prices are necessary for portfolio choice or optimisation, and they may be useful for forecasting future returns on a given portfolio as well. Thus, the n-variate MSF-BEKK model may occur practical and useful also in VaR analysis for large portfolios.

Our empirical study shows that the new hybrid n-variate and univariate models behave quite well and can compete with the CAViaR nonparametric specification. They are important all-purpose alternatives to non-parametric models that were designed to focus on specific aspects of future returns (and not on their full predictive distribution). Note that our univariate hybrid model appears as an interesting by-product of the multivariate analysis. It is a new parametric model that integrates flexibility of the basic SV structure and simplicity of the GARCH(1,1) specification. However, our results suggest that the GARCH part may be unnecessary when the posterior distribution of its parameters is not sharp enough as to exclude zero values. A formal comparison between the pure SV and hybrid SV-GARCH(1,1) models would require calculating the Bayes factor, which is beyond the scope of this paper.

Finally, the paper indicates that the Bayesian approach to VaR analysis is fully relevant and practical. Remind that conditioning on observed data as well as inference on non-linear functions of unobserved quantities (future logarithmic returns) are necessary for any appropriate VaR analysis. Both are natural and easy within Bayesian statistics, equipped with the Markov Chain Monte Carlo (MCMC) simulation tools.

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References

[1] Artzner P., Delbaen F., Eber J.-M., Heath D., (1999), Coherent measures of risk, *Mathematical Finance* 9, 203-228.

[2] Bauwens L., Laurent S., Rombouts J.V.K., (2006), Multivariate GARCH models: A survey, *Journal of Applied Econometrics* 21, 79-109.

[3] Engle R., Manganelli S., (2004), CAViaR: conditional autoregressive Value at Risk by regression quantiles, *Journal of Business and Economic Statistics* 22, 367-381.

[4] Lee T. H., (2008), Loss Functions in Time Series Forecasting, [in:] *International Encyclopedia of the Social Sciences*, 2nd edition. Vol. 4, Macmillan Reference USA, Detroit.

[5] Lopez J.A., (1998), Testing your risk tests, *The Financial Survey*, May–June, 18-20.

[6] O'Hagan A., (1994), *Bayesian Inference*, Edward Arnold, London.

[7] Osiewalski J., (2009), New hybrid models of multivariate volatility (a Bayesian perspective), *Przegląd Statystyczny (Statistical Review)* 56, 15-22.

[8] Osiewalski J., Pajor A., (2009), Bayesian analysis for hybrid MSF-SBEKK models of multivariate volatility, *Central European Journal of Economic Modelling and Econometrics* 1, 179-202.

[9] Pajor A., (2005), Bayesian comparison of bivariate SV models for two related time series, *Acta Universitatis Lodziensis – Folia Oeconomica* 190, 177-196.

[10] Sarma M., Thomas S., Shah A., (2003), Selection of Value-at-Risk Models, *Journal of Forecasting* 22, 337-358.

[11] Tsay R.S., (2005), *Analysis of Financial Time Series* (2nd edition), Wiley, New York.

Appendix

Table A1: Sample characteristics for the first dataset (January 30, 2003 – August 29, 2007; $n = 34$)

Number	company	average	variance	kurtosis	minimum	maximum
1	BPH	0.111	3.633	5.566	-10.566	9.444
2	BDX	0.099	5.738	10.848	-10.807	21.035
3	DUD	0.142	7.464	84.896	-47.505	12.936
4	ECH	0.205	3.570	6.588	-8.278	8.961
5	EMP	0.207	6.703	74.053	-15.575	43.621
6	GRJ	0.187	4.741	10.388	-12.516	15.453
7	BHW	0.054	2.506	28.503	-20.096	8.734
8	BSK	0.096	1.883	7.090	-6.432	6.652
9	KTY	0.112	3.760	5.918	-11.823	9.019
10	KPX	0.318	11.768	19.581	-15.082	35.398
11	KRB	0.048	3.423	20.872	-21.472	8.961
12	MCI	0.370	13.946	11.538	-20.373	33.178
13	MIL	0.130	5.004	9.131	-12.783	14.458
14	MSX	0.160	13.792	12.555	-24.381	28.768
15	MSZ	0.227	18.058	8.234	-25.300	23.974
16	NET	0.021	3.757	16.142	-20.567	8.444
17	EMF	0.093	9.001	15.841	-22.012	24.686
18	ORB	0.126	4.192	7.993	-15.558	10.178
19	PGF	0.101	4.327	16.119	-10.536	21.767
20	PRC	0.027	24.506	11.560	-28.768	34.484
21	STX	0.099	14.130	12.677	-29.523	23.863
22	STP	0.395	7.523	11.815	-9.237	23.309
23	VST	0.325	7.615	9.840	-10.536	18.666
24	AGO	0.007	4.281	5.500	-11.955	8.072
25	BRE	0.167	3.594	5.007	-7.633	8.898
26	BZW	0.109	4.270	4.090	-8.259	7.496
27	CST	0.193	3.802	9.514	-10.488	13.262
28	GTN	0.208	11.946	35.008	-45.392	24.613
29	KGH	0.182	6.471	5.679	-15.590	9.093
30	PEO	0.086	3.854	4.759	-6.579	11.919
31	PKN	0.100	3.610	3.893	-9.298	7.746
32	PXM	0.349	7.723	7.441	-11.725	16.252
33	PND	0.247	16.735	34.983	-53.870	28.395
34	TPS	0.045	3.237	3.731	-8.359	5.617

Table A2: Sample characteristics for the first dataset (May 13, 2005 – February 23, 2010, $n = 50$)

Number	company	average	variance	kurtosis	minimum	maximum	a_i
1	HANDLOWY	0.007	4.854	11.292	-20.096	9.225	1.413
2	INGBSK	0.037	4.897	6.856	-11.647	9.531	0.244
3	NETIA	0.016	3.792	6.764	-10.110	9.531	21.339
4	LPP	0.072	6.020	7.223	-12.234	17.300	0.073
5	STALPROD	0.164	7.635	5.399	-10.882	14.618	0.163
6	SWIECIE	0.033	4.885	6.807	-11.123	12.925	1.560
7	MILLENNIUM	0.026	9.748	6.466	-16.190	14.458	29.605
8	EMPERIA	0.068	7.473	59.876	-18.232	43.621	1.528
9	EUROCASH	0.129	6.005	5.349	-8.224	12.260	6.872
10	KETY	0.002	5.228	6.380	-12.604	12.047	0.992
11	AMREST	0.094	6.069	6.126	-10.821	11.588	1.466
12	ECHO	0.049	8.247	5.948	-11.778	15.498	24.006
13	CCC	0.109	5.618	5.211	-11.584	9.858	1.925
14	BUDIMEX	0.043	6.849	8.383	-10.807	21.035	1.019
15	ELBUDOWA	0.165	5.547	5.456	-8.895	14.041	0.499
16	ORBIS	0.030	6.583	7.285	-15.558	14.497	1.683
17	SYGNITY	-0.165	9.074	8.915	-19.776	21.481	3.608
18	MOSTALWAR	0.209	7.146	6.704	-14.559	16.380	1.353
19	KOGENERA	0.096	5.282	10.381	-13.976	18.623	1.049
20	PEP	0.131	6.179	10.309	-19.980	14.914	2.619
21	NFIEMF	0.079	10.747	10.251	-18.447	24.686	6.821
22	MCI	0.072	15.610	10.049	-20.373	33.178	15.936
23	CIECH	0.020	6.529	9.109	-17.313	13.604	2.203
24	KOPEX	0.148	11.698	9.716	-15.763	28.174	4.014
25	POLNORD	0.105	18.393	27.961	-53.870	28.395	2.304
26	ALCHEMIA	0.126	11.403	15.487	-19.863	30.295	10.777
27	MOSTALZAB	0.151	14.937	6.727	-15.894	23.974	17.707
28	VISTULA	-0.003	11.648	9.974	-24.512	18.232	58.976
29	GANT	0.215	29.602	15.746	-51.975	33.547	2.610
30	IMPEXMET	0.034	11.864	9.416	-14.542	25.131	45.394
31	STALEXP	0.003	13.214	9.945	-21.337	26.065	48.013
32	DUDA	-0.182	15.286	33.898	-47.505	22.314	79.680
33	MOL	0.000	8.952	8.561	-18.232	17.869	0.408
34	KREDYTB	0.041	5.561	37.723	-33.024	13.414	11.703
35	AGORA	-0.081	7.534	5.656	-16.919	10.851	4.746
36	PEKAO	0.019	8.040	6.733	-20.585	13.556	0.635
37	KGHM	0.098	11.039	7.336	-23.624	17.693	1.101
38	PKNORLEN	-0.023	6.404	4.637	-12.158	12.866	2.675
39	PKOBP	0.026	6.176	4.736	-12.223	9.973	2.623
40	TPSA	-0.014	4.092	3.964	-9.022	8.080	4.327
41	BZWBK	0.053	7.845	4.225	-12.143	11.030	0.776
42	ASSECOPOL	0.070	5.295	9.483	-19.506	13.384	1.443
43	GETIN	0.083	7.456	8.714	-14.957	19.479	12.825
44	GTC	0.055	9.773	5.939	-14.660	17.280	4.115
45	TVN	0.043	7.151	6.331	-15.932	12.859	6.840
46	BRE	0.054	8.092	6.212	-14.150	12.900	0.477
47	PBG	0.110	5.341	4.875	-10.003	9.278	0.360
48	POLIMEXMS	0.105	7.982	5.608	-11.725	14.537	19.973
49	CERSANIT	0.028	8.836	5.328	-13.453	13.573	5.467
50	BIOTON	-0.037	14.935	6.687	-16.705	20.479	277.406